

On Curvature in Noncommutative Geometry

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Abstract: A general definition of a bimodule connection in noncommutative geometry has been recently proposed. For a given algebra this definition is compared with the ordinary definition of a connection on a left module over the associated enveloping algebra. The corresponding curvatures are also compared.

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1 Introduction and motivation

Recently a general definition has been given (Mourad 1995, Dubois-Violette *et al.* 1995) of a linear connection in the context of noncommutative geometry which makes essential use of the full bimodule structure of the differential forms. A preliminary version of the curvature of the connection was given (Madore *et al.* 1995) which had the drawback of not being in general a linear map with respect to the right-module structure. It is in fact analogous to the curvature which is implicitly used by those authors (Chamseddine *et al.* 1993, Sitarz 1994, Klimčík *et al.* 1994, Landi *et al.* 1994,) who define a linear connection using the formula for a covariant derivative on an arbitrary left (or right) module (Karoubi 1981, Connes 1986). Our purpose here is to present a modified definition of curvature which is bilinear. Let \mathcal{A} be a general associative algebra (with unit element). This is what replaces in noncommutative geometry the algebra of smooth functions on a smooth (compact) manifold which is used in ordinary differential geometry. By ‘bilinear’ we mean, here and in what follows, bilinear with respect to \mathcal{A} . In fact we shall present two definitions of curvature. The first is valid in all generality and reduces to the ordinary definition of curvature in the commutative case. The second definition seems to be better adapted to ‘extreme’ noncommutative cases, such as the one considered in Section 5.

The definition of a connection as a covariant derivative was given an algebraic form in the Tata lectures by Koszul (1960) and generalized to noncommutative geometry by Karoubi (1981) and Connes (1986, 1994). We shall often use here the expressions ‘connection’ and ‘covariant derivative’ synonymously. In fact we shall distinguish three different types of connections. A ‘left \mathcal{A} -connection’ is a connection on a left \mathcal{A} -module; it satisfies a left Leibniz rule. A ‘bimodule \mathcal{A} -connection’ is a connection on a general bimodule \mathcal{M} which satisfies a left and right Leibniz rule. In the particular case where \mathcal{M} is the module of 1-forms we shall speak of a ‘linear connection’. The precise definitions are given below. A bimodule over an algebra \mathcal{A} is also a left module over the tensor product $\mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^{\text{op}}$ of the algebra with its ‘opposite’. So a bimodule can have a bimodule \mathcal{A} -connection as well as a left \mathcal{A}^e -connection. These two definitions are compared in Section 2. In Section 3 we discuss the curvature of a bimodule connection. In Section 4 we consider an algebra of forms based on derivations and we compare the left connections with the linear connections. We show that in a sense to be made precise the two definitions yield the same bilinear curvature. That is, the extra restriction which the bimodule structure seems to place on the linear connections does not in fact restrict the corresponding curvature. In Section 5 we consider a more abstract geometry whose differential calculus is not based on derivations. In Section 6 a possible definition is given of the curvature of linear connections on braided-commutative algebras. In Section 7 we examine the (left) projective structure of the 1-forms of the Connes-Lott model.

Let \mathcal{A} be an arbitrary algebra and $(\Omega^*(\mathcal{A}), d)$ a differential calculus over \mathcal{A} . One defines a left \mathcal{A} -connection on a left \mathcal{A} -module \mathcal{H} as a covariant derivative

$$\mathcal{H} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \quad (1.1)$$

which satisfies the left Leibniz rule

$$D(f\psi) = df \otimes \psi + fD\psi \quad (1.2)$$

for arbitrary $f \in \mathcal{A}$. This map has a natural extension

$$\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \xrightarrow{\nabla} \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \quad (1.3)$$

given, for $\psi \in \mathcal{H}$ and $\alpha \in \Omega^n(\mathcal{A})$, by $\nabla\psi = D\psi$ and

$$\nabla(\alpha\psi) = d\alpha \otimes \psi + (-1)^n \alpha \nabla\psi.$$

The operator ∇^2 is necessarily left-linear. However when \mathcal{H} is a bimodule it is not in general right-linear.

A covariant derivative on the module $\Omega^1(\mathcal{A})$ must satisfy (1.2). But $\Omega^1(\mathcal{A})$ has also a natural structure as a right \mathcal{A} -module and one must be able to write a corresponding right Leibniz rule in order to construct a bilinear curvature. Quite generally let \mathcal{M} be an arbitrary bimodule. Consider a covariant derivative

$$\mathcal{M} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M} \quad (1.4)$$

which satisfies both a left and a right Leibniz rule. In order to define a right Leibniz rule which is consistent with the left one, it was proposed by Mourad (1995), by Dubois-Violette & Michor (1995) and by Dubois-Violette & Masson (1995) to introduce a generalized permutation

$$\mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\sigma} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}. \quad (1.5)$$

The right Leibniz rule is given then as

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f \quad (1.6)$$

for arbitrary $f \in \mathcal{A}$ and $\xi \in \mathcal{M}$. The purpose of the map σ is to bring the differential to the left while respecting the order of the factors. It is necessarily bilinear (Dubois-Violette *et al.* 1995). We define a bimodule \mathcal{A} -connection to be the couple (D, σ) .

If in particular

$$\mathcal{M} = \Omega^1(\mathcal{A}) \quad (1.7)$$

then we shall refer to the bimodule \mathcal{A} -connection as a linear connection. Although we shall here be concerned principally with this case we shall often consider more general situations. In any case we shall use the more general notation to be able to distinguish the two copies of $\Omega^1(\mathcal{A})$ on the right-hand side of (1.4).

Let $\Omega_u^*(\mathcal{A})$ be the universal differential calculus. Dubois-Violette & Masson (1995) have shown that given an arbitrary left connection on a bimodule \mathcal{M} there always exists a bimodule homomorphism

$$\mathcal{M} \otimes_{\mathcal{A}} \Omega_u^1(\mathcal{A}) \xrightarrow{\sigma(D)} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$$

such that

$$D(\xi f) = \sigma(D)(\xi \otimes d_u f) + (D\xi)f.$$

The notation $\sigma(D)$ is taken from the definition of the symbol of a differential operator. The condition (1.6) means then that $\sigma(D)$ factorizes as a composition of a σ as above and the canonical homomorphism of $\mathcal{M} \otimes_{\mathcal{A}} \Omega_u^1(\mathcal{A})$ onto $\mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$.

Using σ one can also construct (Mourad 1995) an extension

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \quad (1.8)$$

It can also be proved in fact (Bresser *et al.* 1995) that this extension implies the existence of σ . The operator D^2 is not in general left-linear. However if we define π to be the product in $\Omega^*(\mathcal{A})$ and set $\pi_{12} = \pi \otimes 1$ then $\pi_{12}D^2$ is left-linear,

$$\pi_{12}D^2(f\xi) = f\pi_{12}D^2\xi, \quad (1.9)$$

provided the torsion vanishes and the map σ satisfies the condition

$$\pi \circ (\sigma + 1) = 0. \quad (1.10)$$

The map ∇ is related to D on $\mathcal{H} = \Omega^1(\mathcal{A})$ by

$$\nabla^2 = \pi_{12} \circ D^2. \quad (1.11)$$

The left-hand side of this equation is define for a general \mathcal{A} -connection whereas the right-hand side is defined only in the case of a linear connection.

The torsion T is defined to be the map

$$T = d - \pi \circ D \quad (1.12)$$

from Ω^1 into Ω^2 . The restriction (1.7) is here essential. It follows from the condition (1.10) that T is bilinear. A metric can be defined and it can be required to be symmetric using the map σ . The standard condition that the covariant derivative be metric-compatible can be also carried over to the noncommutative case. For more details we refer, for example, to Madore *et al.* (1995).

2 The bimodule structure

For any algebra \mathcal{A} the enveloping algebra \mathcal{A}^e is defined to be

$$\mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^{\text{op}}.$$

A bimodule \mathcal{M} can also be considered then as a left \mathcal{A}^e -module. The differential calculus $\Omega^*(\mathcal{A})$ has a natural extension to a differential calculus $\Omega^*(\mathcal{A}^e)$ given by

$$\Omega^*(\mathcal{A}^e) = \Omega^*(\mathcal{A}) \otimes \Omega^*(\mathcal{A}^{\text{op}}) = (\Omega^*(\mathcal{A}))^e \quad (2.1)$$

with $d(a \otimes b) = da \otimes b + a \otimes db$. This is not the only choice. For example if $\Omega^*(\mathcal{A})$ were the universal calculus over \mathcal{A} then $\Omega^*(\mathcal{A}^e)$ would not be equal to the universal calculus over \mathcal{A}^e . Suppose that \mathcal{M} has a left \mathcal{A}^e -connection

$$\mathcal{M} \xrightarrow{D^e} \Omega^1(\mathcal{A}^e) \otimes_{\mathcal{A}^e} \mathcal{M}. \quad (2.2)$$

From the equality

$$\Omega^1(\mathcal{A}^e) = (\Omega^1(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{A}^{\text{op}}) \oplus (\mathcal{A} \otimes_{\mathbb{C}} \Omega^1(\mathcal{A}^{\text{op}})). \quad (2.3)$$

and using the identification

$$(\mathcal{A} \otimes_{\mathbb{C}} \Omega^1(\mathcal{A}^{\text{op}})) \otimes_{\mathcal{A}^e} \mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \quad (2.4)$$

given by

$$(1 \otimes \xi) \otimes \eta \mapsto \eta \otimes \xi$$

we find that we have

$$\Omega^1(\mathcal{A}^e) \otimes_{\mathcal{A}^e} \mathcal{M} = (\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}) \oplus (\mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})). \quad (2.5)$$

The covariant derivative D^e splits then as the sum of two terms

$$D^e = D_L + D_R. \quad (2.6)$$

From the identifications it is obvious that D_L (D_R) satisfies a left (right) Leibniz rule and is right (left) \mathcal{A} -linear. Such covariant derivatives have been considered by Cuntz & Quillen (1995), by Bresser *et al.* (1995) and by Dabrowski *et al.* (1995).

One can write a (noncommutative) triangular diagram

$$\begin{array}{ccc} & \mathcal{M} & \\ D_L \swarrow & & \searrow D_R \\ \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M} & \xleftarrow{\sigma} & \mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \end{array} \quad (2.7)$$

from which one sees that given an arbitrary bimodule homomorphism (1.5) and a covariant derivative (2.2) one can construct a covariant derivative (1.4) by the formula

$$D = D_L + \sigma \circ D_R \quad (2.8)$$

which satisfies both (1.2) and (1.6).

Suppose further that the differential calculus is such that the differential d of an element $f \in \mathcal{A}$ is of the form

$$df = -[\theta, f], \quad (2.9)$$

for some element $\theta \in \Omega^1$. Then obviously particular choices for D_L and D_R are the expressions

$$D_L \xi = -\theta \otimes \xi, \quad D_R \xi = \xi \otimes \theta. \quad (2.10)$$

Let τ be a bimodule homomorphism from \mathcal{M} into $\Omega^1(\mathcal{A}^e) \otimes_{\mathcal{A}^e} \mathcal{M}$ and decompose

$$\tau = \tau_L + \tau_R \quad (2.11)$$

according to the decomposition (2.5). The most general D_L and D_R are of the form

$$D_L \xi = -\theta \otimes \xi + \tau_L(\xi), \quad D_R \xi = \xi \otimes \theta + \tau_R(\xi). \quad (2.12)$$

Using (2.8) we can construct a covariant derivative

$$D \xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta) \quad (2.13)$$

from (2.10). In Section 4 we shall study a differential calculus for which this is the only possible D .

From the Formula (2.9) we know that there is a bimodule projection of \mathcal{A}^e onto $\Omega^1(\mathcal{A})$. Suppose that $\Omega^1(\mathcal{A})$ is a projective bimodule and let P be the corresponding

projector. We can identify then $\Omega^1(\mathcal{A})$ as a sub-bimodule of the free \mathcal{A}^e -module of rank 1:

$$\Omega^1(\mathcal{A}) = \mathcal{A}^e P.$$

A left \mathcal{A}^e -connection on \mathcal{A}^e as a left \mathcal{A}^e -module is a covariant derivative of the form (1.4) with $\mathcal{M} = \mathcal{A}^e$. The ordinary differential d^e on \mathcal{A}^e ,

$$\mathcal{A}^e \xrightarrow{d^e} \Omega^1(\mathcal{A}^e), \quad (2.14)$$

is clearly a covariant derivative in this sense. The right-hand side can be written using (2.5) as

$$\Omega^1(\mathcal{A}^e) = (\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^e) \oplus (\mathcal{A}^e \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})) \quad (2.15)$$

and so we can split d^e as the sum of two terms d_L and d_R . Let $a \otimes b$ be an element of \mathcal{A}^e . Then we have

$$d_L(a \otimes b) = -[\theta, a] \otimes b = -\theta \otimes (a \otimes b) + (a \otimes b)(\theta \otimes 1). \quad (2.16)$$

In the first term on the right-hand side the first tensor product is over the algebra and the second is over the complex numbers; in the second term the first tensor product is over the complex numbers and the second is over the algebra.

A general element of $\Omega^1(\mathcal{A})$ can be written as a sum of elements of the form $\xi = (a \otimes b)P = aPb$. We have then

$$d_L \xi = -\theta \otimes \xi + \xi(\theta \otimes 1).$$

Define D_L by

$$D_L \xi = (d_L \xi)P. \quad (2.17)$$

Then we obtain the first of equations (2.12) with

$$\tau_L(\xi) = \xi(\theta \otimes P). \quad (2.18)$$

Here, on the right-hand side, the tensor product is over the algebra and $\theta \otimes P$ is an element of $\Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$. This is a left \mathcal{A}^e -module. Similarly one can construct a D_R and a D^e by equation (2.6):

$$D^e \xi = (d^e \xi)P. \quad (2.19)$$

In the case of ordinary geometry with \mathcal{A} equal to the algebra $\mathcal{C}^\infty(V)$ of smooth functions on a smooth manifold V the algebra \mathcal{A}^e is the algebra of smooth functions in two variables. If $\Omega^*(\mathcal{A})$ is the algebra of de Rham differential forms the only possible σ is the permutation and the left and right Leibniz rules are identical. In this case D^e cannot exist. In fact D_L would satisfy a left Leibniz rule and be left linear since the left and right multiplication are equal. In general let \mathcal{M} be the \mathcal{A} -module of smooth sections of a vector bundle over V . Then \mathcal{M} is a \mathcal{A}^e -module. It is important to notice that although it is projective as an \mathcal{A} -module it is never projective as an \mathcal{A}^e -module since a projective \mathcal{A}^e -module consists of 2-point functions.

3 Curvature

Consider a covariant derivative (1.4) which satisfies the left Leibniz rule (1.2). We can define a right-linear curvature by factoring out in the image of ∇^2 all those elements (J = ‘junk’) which do not satisfy the desired condition. Define J as the vector space

$$J = \left\{ \sum_i (\nabla^2(\xi_i f_i) - \nabla^2(\xi_i) f_i) \mid \xi_i \in \mathcal{M}, f_i \in \mathcal{A} \right\}. \quad (3.1)$$

In fact J is a sub-bimodule of $\Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$. It is obviously a left-submodule. Consider the element $\alpha = \nabla^2(\xi g) - \nabla^2(\xi) g \in J$ and let $f \in \mathcal{A}$. We can write

$$\alpha f = (\nabla^2(\xi g f) - \nabla^2(\xi) g f) - (\nabla^2(\xi g f) - \nabla^2(\xi g) f).$$

Therefore $\alpha f \in J$ and J is also a right submodule.

Let p be the projection

$$\Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{p} \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}/J. \quad (3.2)$$

We shall define the curvature of D as the combined map

$$\text{Curv} = -p \circ \nabla^2. \quad (3.3)$$

In the case of a linear connection we can write

$$\text{Curv} = -p \circ \pi_{12} \circ D^2.$$

By construction Curv is left and right-linear:

$$\text{Curv}(f\xi) = f\text{Curv}(\xi), \quad \text{Curv}(\xi f) = \text{Curv}(\xi)f. \quad (3.4)$$

In the next section we shall present an example which illustrates the role which the right-Leibniz rule (1.6) plays in this construction.

Consider the covariant derivative (2.2). One can define a bilinear curvature as the map

$$\text{Curv}_L = -\nabla_L^2 \quad (3.5)$$

from \mathcal{M} into $\Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$. It is bilinear because by construction it is trivially right-linear. In the case where the differential d is given by (2.9) and D_L is given by (2.10) we find that Curv_L is given by the formula

$$\text{Curv}_L(\xi) = (d\theta + \theta^2) \otimes \xi. \quad (3.6)$$

From this expression it is obvious that Curv_L is right-linear; it is easy to verify directly that it is also left-linear because of the fact that the 2-form $d\theta + \theta^2$ commutes with the elements of \mathcal{A} :

$$[d\theta + \theta^2, f] = d[\theta, f] = -d^2 f = 0. \quad (3.7)$$

The covariant derivative (2.2) has also a bilinear curvature 2-form

$$\text{Curv}^e = -\nabla^{e2} \quad (3.8)$$

which naturally decomposes into three terms all of which are bilinear. One of these terms corresponds to the covariant derivative of Section 1 with σ set equal to zero. It takes its values in a space which can be naturally identified with $\Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$. However because the second action of D^e does not commute with that of σ , the corresponding term of Curv^e does not necessarily coincide with the image of Curv . We shall discuss this in an example in Section 5. The curvature of the particular connection (2.19) can be written in terms of the projector P :

$$\text{Curv}^e \xi = -\nabla^{e^2} \xi = -\xi((d^e P)(d^e P)P). \quad (3.9)$$

The extension of D to the tensor product of n copies of $\Omega^1(\mathcal{A})$ defines a covariant derivative on the left module

$$\mathcal{H} = (\Omega^1(\mathcal{A}))^{\otimes n}. \quad (3.10)$$

The curvature is given by (1.11). In the commutative case and, more generally, in the case of a derivation-based differential calculus this curvature can be expressed in terms of the curvature of the covariant derivative (1.4). For a general differential calculus this will not be the case.

The same remarks can be made concerning the torsion (1.12). In general let π be the product map of $(\Omega^1(\mathcal{A}))^{\otimes n}$ into $\Omega^n(\mathcal{A})$. Then one can also define a module homomorphism

$$(\Omega^1(\mathcal{A}))^{\otimes n} \xrightarrow{T_n} \Omega^{n+1}(\mathcal{A}) \quad (3.11)$$

given by

$$T_n = d\pi - \pi \circ D. \quad (3.12)$$

These maps are all left-module homomorphisms. If $\xi \in \Omega^1(\mathcal{A})$ and $\nu \in (\Omega^1(\mathcal{A}))^{\otimes n}$ then we have

$$T_{n+1}(\xi \otimes \nu) = T_1(\xi) \pi(\nu) - \xi T_n(\nu) + \pi \circ ((\sigma + 1) \otimes 1) \xi \otimes \nabla \nu. \quad (3.13)$$

In order for the last term in the previous equation to vanish it is necessary and sufficient that (1.10) be satisfied. In this case one sees by iteration that the T_n can all be expressed in terms of T_1 and therefore that all of them are bimodule homomorphisms.

4 Linear connections on matrix geometries

As a first example we present the case of the algebra M_n of $n \times n$ matrices (Dubois-Violette *et al.* 1989, 1990) with a differential calculus based on derivations (Dubois-Violette 1988). Let λ_r be a set of generators of the Lie algebra of the special linear group SL_n . Then the derivations $e_r = \text{ad } \lambda_r$ is a basis for the derivations of M_n and the dual 1-forms θ^r commute with the elements of M_n . The set of 1-forms $\Omega^1(M_n)$ is a free left (or right) module of rank $n^2 - 1$. The natural map σ which we shall use is given (Madore *et al.* 1995) by

$$\sigma(\theta^r \otimes \theta^s) = \theta^s \otimes \theta^r. \quad (4.1)$$

Quite generally for any algebra \mathcal{A} with a differential calculus which is based on derivations there is a natural map σ given by a permutation of the arguments in the forms. Let X and Y be derivations. Then one can define σ by

$$\sigma(\xi \otimes \eta)(X, Y) = \xi \otimes \eta(Y, X).$$

A general left M_n -connection can be defined by the covariant derivative

$$D\theta^r = -\omega^r_{st} \theta^s \otimes \theta^t \quad (4.2)$$

with ω^r_{st} an arbitrary element of M_n for each value of the indices r, s, t . We write

$$\omega^r_{st} = \Gamma^r_{st} + J^r_{st} \quad (4.3)$$

where the Γ^r_{st} are proportional to the identity in M_n and the J^r_{st} are trace-free. If we require that the torsion vanish then we have (Madore *et al.* 1995)

$$\Gamma^r_{[st]} = C^r_{st} \quad (4.4)$$

where the C^r_{st} are SL_n structure constants.

If we impose the right Leibniz rule we find that

$$0 = D([f, \theta^r]) = [f, D\theta^r] = -[f, J^r_{st}] \theta^s \otimes \theta^t \quad (4.5)$$

for arbitrary $f \in M_n$ and so we see that if the connection is a linear connection then

$$J^r_{st} = 0. \quad (4.6)$$

Consider now the curvature of the left M_n -connection and write

$$\nabla^2 \theta^r = -\Omega^r_{stu} \theta^t \theta^u \otimes \theta^s. \quad (4.7)$$

Then since the elements of the algebra commute with the generators θ^r we have

$$\nabla^2(\theta^r f) - (\nabla^2 \theta^r) f = \nabla^2(f \theta^r) - (\nabla^2 \theta^r) f = -[f, \Omega^r_{stu}] \theta^t \theta^u \otimes \theta^s. \quad (4.8)$$

Since f is arbitrary it follows then that we have

$$\text{Curv}(\theta^r) = \frac{1}{2} R^r_{stu} \theta^t \theta^u \otimes \theta^s \quad (4.9)$$

where the R^r_{stu} are defined uniquely in terms of the Γ^r_{st} :

$$R^r_{stu} = \Gamma^r_{tp}\Gamma^p_{us} - \Gamma^r_{up}\Gamma^p_{ts} - \Gamma^r_{ps}C^p_{tu}. \quad (4.10)$$

That is, R^r_{stu} does not depend on J^r_{st} .

We conclude then that even had we not required the right Leibniz rule and had admitted an extra term of the form J^r_{st} in the expression for the covariant derivative then we would find that the curvature map Curv would remain unchanged. The extra possible terms are eliminated under the projection p of (3.2).

There is a covariant derivative which is of the form (2.8) with D_L and D_R given by (2.10). For this covariant derivative one has

$$\omega^r_{st} \equiv 0. \quad (4.11)$$

This covariant derivative has obviously vanishing curvature but it is not torsion-free. If we use the ambiguity (2.11) we can write any covariant derivative (4.2) in the form (2.8).

The generators θ^r are no longer independent if one considers the bimodule structure. In fact one finds that

$$\theta^r = -C^r_{st}\lambda^s\theta\lambda^t, \quad \theta = -\lambda_r\theta^r \quad (4.12)$$

and as a bimodule $\Omega^1(M_n)$ is generated by θ alone. For dimensional reasons $\Omega^1(M_n)$ cannot be of rank one. In fact the free M_n -bimodule of rank one is of dimension n^4 and the dimension of $\Omega^1(M_n)$ is equal to $(n^2 - 1)n^2 < n^4$. With the normalization which we have used for the generators λ_r the element

$$\zeta = \frac{1}{n^2}1 \otimes 1 - \frac{1}{n}\lambda_r \otimes \lambda^r$$

is a projector in $M_n \otimes M_n$ which commutes with the elements of M_n . This can be written as

$$d(M_n)\zeta = 0.$$

We have the direct-sum decomposition

$$M_n \otimes M_n = \Omega^1(M_n) \oplus M_n \zeta. \quad (4.13)$$

As in Section 1 one can define $M_n^e = M_n \otimes_{\mathbb{C}} M_n^{\text{op}}$. The prescription (2.19) with

$$P = 1 \otimes 1 - \zeta$$

yields then a covariant derivative of the form (4.2) whose curvature vanishes.

5 Linear connections on the Connes-Lott model

Consider the algebra M_3 with the grading defined by the decomposition $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$. Define (Connes & Lott 1990, 1992) $\Omega^0(M_3^+) = M_3^+ = M_2 \times M_1$, $\Omega^1(M_3^+) = M_3^-$, $\Omega^2(M_3^+) = M_1$ and $\Omega^p(M_3^+) = 0$ for $p \geq 3$. A differential d can be defined by (Connes 1986, 1990)

$$df = -[\theta, f], \quad (5.1)$$

where $\theta \in \Omega^1(M_3^+)$.

The vector space of 1-forms is of dimension 4 over the complex numbers. The dimension of $\Omega^1(M_3^+) \otimes_{\mathbb{C}} \Omega^1(M_3^+)$ is equal to 16 but the dimension of the tensor product $\Omega^1(M_3^+) \otimes_{M_3^+} \Omega^1(M_3^+)$ is equal to 5 and we can make the identification

$$\Omega^1(M_3^+) \otimes_{M_3^+} \Omega^1(M_3^+) = M_3^+. \quad (5.2)$$

To define a linear connection we must first define the map σ of (1.5) with $\mathcal{M} = \Omega^1(M_3^+)$. Because of the identification (5.2) it can be considered as a map from M_3^+ into itself and because of the bilinearity it is necessarily of the form

$$\sigma = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $\mu \in \mathbb{C}$. The -1 in the lower right corner is imposed by the condition (1.10).

It can be shown (Madore *et al.* 1995) that for each such σ there is a unique linear connection given by the covariant derivative (2.13). That is, necessarily $\tau \equiv 0$. Let e be the unit in M_1 considered as generator of $\Omega^2(M_3^+)$. The expression $d\theta + \theta^2$ is given by

$$d\theta + \theta^2 = e.$$

Therefore we have

$$\text{Curv}_L(\xi) = e \otimes \xi. \quad (5.3)$$

To construct J it is convenient to fix a vector-space basis for $\Omega^1(M_3^+)$. We introduce the (unique) upper-triangular matrices η_1 and η_2 such that $\theta = \eta_1 - \eta_1^*$ and such that

$$\eta_i \eta_j^* = 0, \quad \eta_i^* \eta_j = \delta_{ij} e.$$

We find (Madore *et al.* 1995) that

$$\begin{aligned} \nabla^2 \eta_1 &= 0, & \nabla^2 \eta_2 &= 0, \\ \nabla^2 \eta_1^* &= -(\mu + 1)e \otimes \eta_1^*, & \nabla^2 \eta_2^* &= -e \otimes \eta_2^*. \end{aligned} \quad (5.4)$$

Since there is an element u of the algebra such that $\eta_2 = u\eta_1$ it is obvious that the map ∇^2 is right-linear only in the degenerate case $\mu = 0$. In this case $J = 0$ and

$$\text{Curv} = \text{Curv}_L. \quad (5.5)$$

Otherwise it is easy to see that

$$J = \Omega^2(M_3^+) \otimes_{M_3^+} \Omega^1(M_3^+)$$

and therefore that

$$\text{Curv} \equiv 0. \quad (5.6)$$

It is difficult to appreciate the meaning of this result since there is only one connection for each value of μ . However (5.4) does not appear to define a curvature which is any the less flat for generic μ than for the special value $\mu = 0$. As we have defined it Curv does not perhaps contain enough information to characterize a general noncommutative geometry.

From (5.4) one sees that for all values of μ there is a subalgebra of M_3^+ with respect to which ∇^2 is right-linear. It consists of those elements which leave invariant the vector sub-spaces of $\Omega_1(M_3^+)$ defined by η_1 and η_2 . That is, it is the algebra

$$M_1 \times M_1 \times M_1 \subset M_3^+.$$

As in Section 1 we define $M_3^{+e} = M_3^+ \otimes_{\mathbb{C}} M_3^{+\text{op}}$. A general element ξ of $\Omega^1(M_3^+)$ can be written in the form

$$\xi = \begin{pmatrix} 0 & 0 & \xi_{13} \\ 0 & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{pmatrix}, \quad (5.7)$$

where the ξ_{ij} are arbitrary complex numbers. The map

$$\xi \mapsto \begin{pmatrix} 0 & \xi_{13} & 0 \\ 0 & \xi_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ \xi_{31} & \xi_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.8)$$

identifies $\Omega^1(M_3^+)$ as a sub-bimodule of the free M_3^{+e} -module of rank 1 and the projector

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.9)$$

projects M_3^{+e} onto $\Omega^1(M_3^+)$. One immediately sees that by multiplication of P on the right and left by elements of M_3^+ one obtains all elements of $\Omega^1(M_3^+)$. The construction of Section 2 can be used to construct by projection a covariant derivative (2.19). In the present case we find that

$$P(\theta \otimes P) = 0$$

as it must be since we have already noticed that in the present case $\tau \equiv 0$. The covariant derivative (2.19) is identical to that given by (2.10).

6 Braided-commutative algebras

As an example of a braided-commutative differential calculus we consider the quantum plane with its $SL_q(2)$ -covariant differential calculus Ω^* . It has been found (Dubois-Violette *et al.* 1995) that there is a unique 1-parameter family of linear connections given by the covariant derivative

$$D\xi = \mu^4 x^i (x\eta - qy\xi) \otimes (x\eta - qy\xi), \quad (6.1)$$

with μ a complex number. The corresponding σ is uniquely defined in terms of the R -matrix. There are other linear connections if we extend the algebra to include additional elements x^{-1} and y^{-1} . For example consider the construction of Section 2 based on the formula (2.8). For arbitrary complex number c define for $q \neq 1$

$$\theta = \frac{1}{1 - q^{-2}} (y^{-1}\eta + cxy^{-2}\xi). \quad (6.2)$$

We have then

$$\xi^i = dx^i = -[\theta, x^i]. \quad (6.3)$$

If $c = 0$ then the differential d is given on the entire algebra of forms as a graded commutator with θ . Using θ we can define D_L and D_R by (2.10) and a covariant derivative by (2.13). As in Section 4 the curvature of this covariant derivative vanishes. In fact for arbitrary c we have $d\theta + \theta^2 = 0$. This construction can be used for any generalized permutation which satisfies the condition (1.10). There are many such σ . For example if i is a bimodule injection of Ω^2 into $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ which satisfies the condition $\pi \circ i = 1$ then a generalized permutation is given by the formula $\sigma = 1 - 2i \circ \pi$ (Mourad 1995).

A linear connection has also been constructed on $GL_q(n)$ (Georgelin *et al.* 1995). The differential calculus is constructed using a 1-form θ and a linear connection is given by the Formula (2.13).

The construction of a bilinear curvature based on the projection (3.2) is not interesting in the general braided-commutative case. In this case the right-module structure of $\Omega^1(\mathcal{A})$ is determined in terms of its left-module structure even though the forms do not commute with the algebra. The construction can be modified however using the braiding. There is then a morphism ρ of the algebra such that the vector space

$$J_\rho = \left\{ \sum_i (\nabla^2(\xi_i f_i) - \nabla^2(\xi_i) \rho(f_i)) \mid \xi_i \in \Omega^1(\mathcal{A}), f_i \in \mathcal{A} \right\}$$

vanishes identically. The curvature Curv_ρ defined by the obvious modification of (3.3) is therefore left linear and right ρ -linear:

$$\text{Curv}_\rho(f\xi) = f\text{Curv}_\rho(\xi), \quad \text{Curv}_\rho(\xi f) = \text{Curv}_\rho(\xi)\rho(f). \quad (6.4)$$

In general, for each automorphism ρ of the algebra, a curvature Curv_ρ can be defined; it would be of interest however only in the case when J_ρ vanishes.

7 The problem of curvature invariants

In Section 4 we considered a geometry with a module of 1-forms which was free of rank $n^2 - 1$ as a left (and right) module. We noticed also, in (4.13) that it can be written as a direct summand in a free bimodule of rank 1. In Section 5 we considered the projective bimodule structure of $\Omega^1(M_3^+)$. In this Section we shall examine the projective structure of $\Omega^1(M_3^+)$ as a left (and right) module in order to see to what extent it is possible to express the geometry of Section 5 in the language of Section 4. If it were possible to do this it would be possible to define curvature invariants as in Section 4.

Consider the element (5.7) of $\Omega^1(M_3^+)$. The map

$$\xi \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_{31} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_{32} \end{pmatrix}, \quad \begin{pmatrix} 0 & \xi_{13} & 0 \\ 0 & \xi_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7.5)$$

identifies $\Omega^1(M_3^+)$ as a submodule of the free module

$$\mathcal{M} \equiv (M_3^+)^3 = M_3^+ \oplus M_3^+ \oplus M_3^+$$

of rank 3. This imbedding respects the left-module structure of M_3^+ . It respects also the right-module structure if we identify

$$f \mapsto \rho(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes f. \quad (7.6)$$

That is, under the right action by M_3^+ the element f it acts on the row vector and not on the matrix entries.

Define the projectors

$$P_1 = P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7.7)$$

in M_3^+ and the projector

$$P = P_1 \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + P_2 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + P_3 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7.8)$$

in $M_3(M_3^+)$. Then $\xi P = \xi$. Let α be a general element of \mathcal{M} , that is, a triplet of elements of M_3^+ written as in (7.5) as a row vector. Then $\alpha P \in \Omega^1(M_3^+)$ and all elements $\xi \in \Omega^1(M_3^+)$ can be obtained in this way; the module $\Omega^1(M_3^+)$ is a projective left M_3^+ -module:

$$\Omega^1(M_3^+) = \mathcal{M}P. \quad (7.9)$$

This defines a projection

$$\mathcal{M} \xrightarrow{P} \Omega^1(M_3^+) \quad (7.10)$$

which is a left inverse of the imbedding (7.5).

Let θ^r be the canonical basis of \mathcal{M} :

$$\theta^1 = (1, 0, 0), \quad \theta^2 = (0, 1, 0), \quad \theta^3 = (0, 0, 1)$$

where the unit is the unit in M_3^+ . We use a notation here which parallels that of Section 4 (with $n = 2$). In general however, for $f \in M_3^+$,

$$f\theta^r \neq \theta^r \rho(f) \equiv \theta^s (\rho(f))_s^r.$$

This is an essential difference with the geometry of Section 4.

Define

$$\theta_P^r = \theta^r P.$$

By this we mean that θ_P^r is the image of the triplet θ^r under the projection (7.10) which we again identify as an element of \mathcal{M} by (7.5). An extension $\tilde{\sigma}$ of σ is a map

$$\Omega^1(M_3^+) \otimes_{M_3^+} \mathcal{M} \xrightarrow{\tilde{\sigma}} \mathcal{M} \otimes_{M_3^+} \Omega^1(M_3^+)$$

given by the action $\tilde{\sigma}(\theta_P^r \otimes \theta^s)$. It is clear that this $\tilde{\sigma}$ will not be a simple permutation as in (4.1). The covariant derivative on \mathcal{M} will be defined by

$$\tilde{D}\theta^r = -\omega^r_{st} \theta_P^s \otimes \theta^t \quad (7.11)$$

analogous to (4.2) but with here the ω^r_{st} arbitrary elements of M_3^+ .

Using the projection p we can define in particular a covariant derivative \tilde{D} on \mathcal{M} which coincides with the image of D on $\Omega^1(M_3^+)$ by the requirement that the diagram

$$\begin{array}{ccc} \Omega^1(M_3^+) & \xrightarrow{D} & \Omega^1(M_3^+) \otimes_{M_3^+} \Omega^1(M_3^+) \\ p \uparrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\tilde{D}} & \Omega^1(M_3^+) \otimes_{M_3^+} \mathcal{M} \end{array} \quad (7.12)$$

be commutative. The down arrow on the right is an injection defined by (7.5). The covariant derivative $\tilde{D}\theta^r$ is defined then by

$$\tilde{D}\theta^r = D\theta_P^r. \quad (7.13)$$

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